

# Lie Bialgebra Structures on Generalized Heisenberg-Virasoro Algebra<sup>1</sup>

Hai Bo Chen<sup>1)</sup>, Ran Shen<sup>2)</sup>, Jian Gang Zhang<sup>3)</sup>

<sup>1), 2)</sup> College of Science, Donghua University, Shanghai, 201620, China

<sup>3)</sup> Department of Mathematics, Shanghai Normal University, Shanghai, 200234, China  
Email: rshen@dhu.edu.cn

**Abstract** In this paper, Lie bialgebra structures on generalized Heisenberg-Virasoro algebra  $\mathfrak{L}$  are considered. Also,  $H^1(\mathfrak{L}, \mathfrak{L} \otimes \mathfrak{L})$  is given explicitly. Moreover, it is proved that all Lie bialgebra structure on centerless generalized Heisenberg-Virasoro algebra  $\overline{\mathfrak{L}}$  are coboundary triangular.

**Key words** Lie bialgebras, Yang-Baxter equation, generalized Heisenberg-Virasoro algebra.

**MR(2000) Subject Classification** 17B62, 17B05, 17B37, 17B66

## 1. INTRODUCTION

Lie bialgebra structures on some Lie (super)algebras including generalized Witt type, generalized Virasoro like type and generalized Weyl type Lie algebras, the Schrödinger-Virasoro Lie algebra, the  $N = 2$  superconformal algebra, etc., were constructed (cf. [4], [6], [7], [10], [12], [13], [14], [15], [16]) since the notion was first introduced by Drinfeld in 1983 (cf. [1], [2]) in a connection with quantum groups. Recently, a general method to obtain Lie (super)bialgebra structures on some Lie (super) algebras related to Virasoro algebra was given in [3].

In this paper, We will study the Lie bialgebra structures on Lie algebra of generalized Heisenberg-Virasoro algebra which has been studied in [5]. It is the natural generalization of the twisted Heisenberg-Virasoro algebra. The structure and representations for generalized Heisenberg-Virasoro algebra were studied in [5], [9]. However, Lie bialgebra structures on generalized Heisenberg-Virasoro algebra have not yet been considered.

Let us recall some definitions related to Lie bialgebras. For a vector space  $\mathfrak{L}$  over a field  $\mathbb{F}$  of characteristic zero. we define the *twist map*  $\tau$  of  $\mathfrak{L} \otimes \mathfrak{L}$  and the *cyclic map*  $\xi$  of  $\mathfrak{L} \otimes \mathfrak{L} \otimes \mathfrak{L}$  by

$$(1.1) \quad \tau : x \otimes y \mapsto y \otimes x, \quad \xi : x \otimes y \otimes z \mapsto y \otimes z \otimes x \quad \text{for } x, y, z \in \mathfrak{L}.$$

The definitions of a Lie algebra and Lie coalgebra can be reformulated as follows. Then a *Lie algebra* can be defined as a pair  $(\mathfrak{L}, \varphi)$  consisting of a vector space  $\mathfrak{L}$  and a bilinear map  $\varphi : \mathfrak{L} \otimes \mathfrak{L} \rightarrow \mathfrak{L}$  (called the *bracket* of  $\mathfrak{L}$ ) satisfying the following conditions:

$$(1.2) \quad \text{Ker}(1 - \tau) \subset \text{Ker } \varphi \quad (\text{skew-symmetry}),$$

$$(1.3) \quad \varphi \cdot (1 \otimes \varphi) \cdot (1 + \xi + \xi^2) = 0 : \mathfrak{L} \otimes \mathfrak{L} \otimes \mathfrak{L} \rightarrow \mathfrak{L} \quad (\text{Jacobi identity}),$$

denote  $1$  is the identity map of  $\mathfrak{L} \otimes \mathfrak{L}$ . Dually, a *Lie coalgebra* is a pair  $(\mathfrak{L}, \Delta)$  consisting of a vector space  $\mathfrak{L}$  and a linear map  $\Delta : \mathfrak{L} \rightarrow \mathfrak{L} \otimes \mathfrak{L}$  (called the *cobacket*

---

<sup>1</sup>Supported by NSF of China (No.11001046), the Fundamental Research Funds for the Central Universities, "Outstanding young teachers of Donghua University" foundation.

of  $\mathfrak{L}$ ) satisfying the following conditions:

$$(1.4) \quad \text{Im } \Delta \subset \text{Im}(1 - \tau) \quad (\text{anti-commutativity}),$$

$$(1.5) \quad (1 + \xi + \xi^2) \cdot (1 \otimes \Delta) \cdot \Delta = 0 : \mathfrak{L} \rightarrow \mathfrak{L} \otimes \mathfrak{L} \otimes \mathfrak{L} \quad (\text{Jacobi identity}).$$

*Definition 1.1.* A *Lie bialgebra* is a triple  $(\mathfrak{L}, \varphi, \Delta)$  such that  $(\mathfrak{L}, \varphi)$  is a Lie algebra and  $(\mathfrak{L}, \Delta)$  is a Lie coalgebra and the following *compatibility condition* holds:

$$(1.6) \quad \Delta\varphi(x, y) = x \cdot \Delta y - y \cdot \Delta x \quad \text{for } x, y \in \mathfrak{L},$$

We shall use the symbol “.” to stand for the *diagonal adjoint action*:

$$(1.7) \quad x \cdot (\sum_i a_i \otimes b_i) = \sum_i ([x, a_i] \otimes b_i + a_i \otimes [x, b_i])$$

for  $x, a_i, b_i \in \mathfrak{L}$ , and in general  $[x, y] = \varphi(x, y)$  for  $x, y \in \mathfrak{L}$ .

*Definition 1.2.* (1) A *coboundary Lie bialgebra* is a 4-tuple  $(\mathfrak{L}, \varphi, \Delta, r)$ , where  $(\mathfrak{L}, \varphi, \Delta)$  is a Lie bialgebra and  $r \in \text{Im}(1 - \tau) \subset \mathfrak{L} \otimes \mathfrak{L}$  such that  $\Delta = \Delta_r$  is a *coboundary of  $r$* , where in general  $\Delta_r$  (which is an inner derivation, cf. (1.15)) is defined by,

$$(1.8) \quad \Delta_r(x) = x \cdot r \quad \text{for } x \in \mathfrak{L}.$$

(2) A coboundary Lie bialgebra  $(\mathfrak{L}, \varphi, \Delta, r)$  is *triangular* if it satisfies the following *classical Yang-Baxter Equation* (CYBE):

$$(1.9) \quad c(r) = 0.$$

(3) An element  $r \in \text{Im}(1 - \tau) \subset \mathfrak{L} \otimes \mathfrak{L}$  is said to satisfy the *modified Yang-Baxter Equation* (MYBE) if

$$(1.10) \quad x \cdot c(r) = 0, \quad \text{for } x \in \mathfrak{L}.$$

where  $c(r)$  is defined by

$$(1.11) \quad c(r) = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}],$$

and  $r^{ij}$  are defined as follows: Denote  $\mathcal{U}(\mathfrak{L})$  the universal enveloping algebra of  $\mathfrak{L}$  and 1 the identity element of  $\mathcal{U}(\mathfrak{L})$ . If  $r = \sum_i a_i \otimes b_i \in \mathfrak{L} \otimes \mathfrak{L}$ , then  $r^{ij}$  are the following elements in  $\mathcal{U}(\mathfrak{L}) \otimes \mathcal{U}(\mathfrak{L}) \otimes \mathcal{U}(\mathfrak{L})$ :

$$r^{12} = r \otimes 1 = \sum_i a_i \otimes b_i \otimes 1,$$

$$r^{13} = (1 \otimes \tau)(r \otimes 1) = \sum_i a_i \otimes 1 \otimes b_i,$$

$$r^{23} = 1 \otimes r = \sum_i 1 \otimes a_i \otimes b_i.$$

The following results can be found in [2] and [8].

*Lemma 1.3.* (1) For a Lie algebra  $\mathfrak{L}$  and  $r \in \text{Im}(1 - \tau) \subset \mathfrak{L}$ , the tripple  $(\mathfrak{L}, [\cdot, \cdot], \Delta)$  is a Lie bialgebra if and only if  $r$  satisfies MYBE.

(2) For a Lie algebra  $\mathfrak{L}$  and  $r \in \text{Im}(1 - \tau) \subset \mathfrak{L}$ , we have

$$(1.12) \quad (1 + \xi + \xi^2) \cdot (1 \otimes \Delta) \cdot \Delta(x) = x \cdot c(r) \quad \text{for all } x \in \mathfrak{L}.$$

Let us state our main results below. Suppose  $\Gamma$  be an abelian group and  $T$  is a vector space over  $\mathbb{F}$ . We always assume that  $T = \mathbb{F}\partial$  because of  $\dim T = 1$ . The tensor product  $W = \mathbb{F}\Gamma \otimes_{\mathbb{F}} T$  is free left  $\mathbb{F}\Gamma$ -module. We shall denote  $t^x \partial = t^x \otimes \partial$ . Fix a pairing  $\varphi : T \times \Gamma \rightarrow \mathbb{F}$ , which is  $\mathbb{F}$ -linear in the first variable and additive in the second one. Denote:  $\varphi(\partial, x) = \langle \partial, x \rangle = \partial(x)$  for  $x \in \Gamma$ . If  $\Gamma_0 = \{x \in \Gamma :$

$\partial(x) = 0\} = 0$  called  $\varphi$  is nondegenerate. In this paper, we always suppose that  $\varphi$  is nondegenerate.

Clearly, from  $\Gamma$  and  $T$  is a vector space over  $\mathbb{F}$ , the *generalized Heisenberg-Virasoro algebra*  $\mathfrak{L} := \mathfrak{L}(\Gamma)([5])$  is a Lie algebra generated by  $\{L_x = t^x \partial, I_x = t^x, C_L, C_I, C_{LI}, x \in \Gamma\}$ , subject to the following relations:

$$[L_x, L_y] = (y - x)L_{x+y} + \delta_{x+y,0} \frac{1}{12}(x^3 - x)C_L,$$

$$[I_x, I_y] = y\delta_{x+y,0}C_I,$$

$$[L_x, I_y] = yI_{x+y} + \delta_{x+y,0}(x^2 - x)C_L,$$

$$[\mathfrak{L}, C_L] = [\mathfrak{L}, C_I] = [\mathfrak{L}, C_{LI}] = 0.$$

The Lie algebra  $\mathfrak{L}$  has a generalized Heisenberg subalgebra and a generalized Virasoro subalgebra intertwined with a 2-cocycle. Set  $\mathfrak{L}_x = \text{Span}_{\mathbb{F}}\{L_x, I_x\}$  for  $x \in \Gamma \setminus \{0\}$ ,  $\mathfrak{L}_0 = \text{Span}_{\mathbb{F}}\{I_0, I_0, C_L, C_I, C_{LI}\}$ . Then  $\mathfrak{L} = \bigoplus_{x \in \Gamma} \mathfrak{L}_x$  is a graded Lie algebra. Denote  $\mathcal{C}$  the center of  $\mathfrak{L}$ , then  $\mathcal{C} = \text{Span}_{\mathbb{F}}\{I_0, C_L, C_I, C_{LI}\}$ . It is well known that the first cohomology group of  $\mathfrak{L}$  with coefficients in the module  $V$  is isomorphic to

$$(1.13) \quad H^1(\mathfrak{L}, V) \cong \text{Der}(\mathfrak{L}, V) / \text{Inn}(\mathfrak{L}, V),$$

where  $\text{Der}(\mathfrak{L}, V)$  is the set of *derivations*  $d : \mathfrak{L} \rightarrow V$  which are linear maps satisfying

$$(1.14) \quad d([x, y]) = x \cdot d(y) - y \cdot d(x) \quad \text{for } x, y \in \mathfrak{L},$$

and  $\text{Inn}(\mathfrak{L}, V)$  is the set of *inner derivations*  $a_{\text{inn}}, a \in V$ , defined by

$$(1.15) \quad a_{\text{inn}} : x \mapsto x \cdot a \quad \text{for } x \in \mathfrak{L}.$$

## 2. LIE BIALGEBRA STRUCTURES ON THE GENERALIZED HEISENBERG-VIRASORO ALGEBRA

*Definition 2.1.* For any  $\lambda \in \mathbb{F}, C \in \mathcal{C}$ , we define the map  $\lambda \otimes C : \mathfrak{L} \rightarrow \mathfrak{L} \otimes \mathfrak{L}$  by

$$(2.1) \quad (\lambda \otimes C)(\omega_\alpha) = \lambda(1 - \delta_{\alpha,0})w_1 I_\alpha \otimes C,$$

for  $\omega_\alpha = w_1 L_\alpha + w_2 I_\alpha + \delta_{\alpha,0} Z \in \mathfrak{L}_\alpha$ , where  $Z \in \text{Span}_{\mathbb{F}}\{C_L, C_{LI}, C_I\}, \alpha \in \Gamma$ .

Obviously, for any  $C \in \mathcal{C}, \lambda \in \mathbb{F}, \lambda \otimes C \in \text{Der}(\mathfrak{L}, \mathfrak{L} \otimes \mathfrak{L})$  and it is an outer derivation. Furthermore,  $\mathbb{F} \otimes C = \{\lambda \otimes C, \lambda \in \mathbb{F}\}$  is a subalgebra of  $\text{Der}(\mathfrak{L}, \mathfrak{L} \otimes \mathfrak{L})$ , denoted by  $\mathbb{F} \otimes \mathbb{F}C$ . Similarly, we have the derivation subalgebra  $\mathbb{F}C \otimes \mathbb{F}$ .

The main result of this section is

*Theorem 2.2.* (1) Let  $(\mathfrak{L}, [\cdot, \cdot], \Delta)$  be a Lie bialgebra such that  $\Delta$  has the decomposition  $\Delta_r + \sigma$  with respect to  $\text{Der}(\mathfrak{L}, V) = \text{Inn}(\mathfrak{L}, V) \oplus (\mathbb{F} \otimes \mathbb{F}C + \mathbb{F}C \otimes \mathbb{F})$ , where  $r \in V(\text{mod } \mathcal{C} \otimes \mathcal{C})$  and  $\sigma \in \mathbb{F} \otimes \mathbb{F}C + \mathbb{F}C \otimes \mathbb{F}, \sigma(\mathfrak{L}) \subseteq \text{Im}(1 - \tau)$ . Then,  $r \in \text{Im}(1 - \tau)$ . Furthermore,  $(\mathfrak{L}, [\cdot, \cdot], \sigma)$  is a Lie bialgebra.

(2) An element  $r \in \text{Im}(1 - \tau) \subset \mathfrak{L} \otimes \mathfrak{L}$  satisfies CYBE in (1.9) if and only if it satisfies MYBE in (1.10).

(3) Regarding  $V = \mathfrak{L} \otimes \mathfrak{L}$  as an  $\mathfrak{L}$ -module under the adjoint diagonal action of  $\mathfrak{L}$  in (1.7), we have  $H^1(\mathfrak{L}, V) = \text{Der}(\mathfrak{L}, V) / \text{Inn}(\mathfrak{L}, V) \cong \mathbb{F} \otimes \mathbb{F}C + \mathbb{F}C \otimes \mathbb{F}$ .

We give the proof of Theorem 2.2 by several lemmas and propositions.

At first, Theorem 2.2(2) follows from the following result.

*Lemma 2.3.* Denote by  $\mathfrak{L}^{\otimes n}$  the tensor product of  $n$  copies of  $\mathfrak{L}$ . Regarding  $\mathfrak{L}^{\otimes n}$  as an  $\mathfrak{L}$ -module under the adjoint diagonal action of  $\mathfrak{L}$ , suppose  $c \in \mathfrak{L}^{\otimes n}$  satisfying  $a \cdot c = 0$  for all  $a \in \mathfrak{L}$ . Then  $c = 0$ .

*Proof.* The lemma is obtained by using the same arguments as in the proof of Lemma 2.2 in [13].

Theorem 2.2(3) follows from the following proposition.

*Proposition 2.4.*  $\text{Der}(\mathfrak{L}, V) = \text{Inn}(\mathfrak{L}, V) + \mathbb{F} \otimes \mathbb{F}C + \mathbb{F}C \otimes \mathbb{F}$ , where  $V = \mathfrak{L} \otimes \mathfrak{L}$ .

*Proof.* We shall divide the proof of the proposition into several claims. Note that  $V = \bigoplus_{\alpha \in \Gamma} V_{\alpha}$  is  $\Gamma$ -graded with  $V_{\alpha} = \sum_{\beta+\gamma=\alpha} \mathfrak{L}_{\beta} \otimes \mathfrak{L}_{\gamma}$ , where  $\mathfrak{L}_{\alpha} = \mathbb{C}L_{\alpha} \oplus \mathbb{C}I_{\alpha} \oplus \delta_{\alpha,0}(\mathbb{C}C_L + \mathbb{C}C_I + \mathbb{C}C_{LI})$  for  $\alpha \in \Gamma$ . A derivation  $D \in \text{Der}(\mathfrak{L}, V)$  is *homogeneous of degree  $\alpha \in \Gamma$*  if  $D(V_{\beta}) \subset V_{\alpha+\beta}$  for all  $\beta \in \Gamma$ . Denote  $\text{Der}(\mathfrak{L}, V)_{\alpha} = \{D \in \text{Der}(\mathfrak{L}, V) \mid \deg D = \alpha\}$  for  $\alpha \in \Gamma$ .

*Claim 1.* Every derivation  $D \in \text{Der}(\mathfrak{L}, V)$ . Then

$$(2.2) \quad D = \sum_{\alpha \in \Gamma} D_{\alpha}, \quad \text{where } D_{\alpha} \in \text{Der}(\mathfrak{L}, V)_{\alpha},$$

which holds in the sense that for every  $\omega \in \mathfrak{L}$ , only finitely many  $D_{\alpha}(\omega) \neq 0$ , and  $D(\omega) = \sum_{\alpha \in \Gamma} D_{\alpha}(\omega)$  (we call such a sum in (2.2) *summable*).

For  $\alpha \in \Gamma$ , we define a homogeneous linear map  $D_{\alpha} : \mathfrak{L} \rightarrow V$  of degree  $\alpha$  as follows: For any  $\omega \in \mathfrak{L}_{\beta}$  with  $\beta \in \Gamma$ , write  $d(\omega) = \sum_{\gamma \in \Gamma} v_{\gamma} \in V$  with  $v_{\gamma} \in V_{\gamma}$ , then we set  $D_{\alpha}(\omega) = v_{\alpha+\beta}$ . Obviously  $D_{\alpha} \in \text{Der}(\mathfrak{L}, V)_{\alpha}$  and we have (2.2).

*Claim 2.* If  $0 \neq \alpha \in \Gamma$ , then  $D_{\alpha} \in \text{Inn}(\mathfrak{L}, V)$ .

Denote  $u = \alpha^{-1}D_{\alpha}(L_0) \in V_{\alpha}$ . For any  $\omega_{\beta} \in \mathfrak{L}_{\beta}$ ,  $\beta \in \Gamma$ , applying  $D_{\alpha}$  to  $[L_0, \omega_{\beta}] = \beta\omega_{\beta}$ , using  $D_{\alpha}(\omega_{\beta}) \in V_{\alpha+\beta}$  and the action of  $L_0$  on  $V_{\alpha+\beta}$  is the scalar  $\alpha + \beta$ , we have

$$(2.3) \quad (\alpha + \beta)D_{\alpha}(\omega_{\beta}) - \omega_{\beta} \cdot D_{\alpha}(L_0) = \beta D_{\alpha}(\omega_{\beta}),$$

i.e.,  $D_{\alpha}(\omega_{\beta}) = u_{\text{inn}}(\omega_{\beta})$  (cf. (1.15)). Thus  $D_{\alpha} = u_{\text{inn}}$  is inner.

*Claim 3.*  $D_0(L_0) = D_0(C) = 0 \pmod{\mathbb{F}(\mathcal{C} \otimes \mathcal{C})}$ .

In order to prove this, applying  $D_0$  to  $[L_0, \omega] = \beta\omega$  for  $\omega \in \mathfrak{L}_{\beta}$ ,  $\beta \in \Gamma$ , as in (2.3), we obtain that  $\omega \cdot D_0(L_0) = 0 \pmod{\mathbb{F}(\mathcal{C} \otimes \mathcal{C})}$ . Thus by lemma 2.3,  $D_0(L_0) = 0$ . Similarly, by applying  $D_0$  to  $[C, \omega] = 0$ , we obtain  $D_0(C) = 0 \pmod{\mathbb{F}(\mathcal{C} \otimes \mathcal{C})}$ .

*Claim 4.* By replacing  $D_0$  by  $D_0 - u_{\text{inn}} - (\lambda \otimes C + C \otimes \eta)$  for some  $u \in V_0$ ,  $\lambda, \eta \in \mathbb{F}$ , we can suppose  $D_0(\mathfrak{L}_{\mu}) \equiv 0 \pmod{\mathbb{F}(\mathcal{C} \otimes \mathcal{C})}$  for  $\mu \in \Gamma$ .

We can write, under modulo  $\mathbb{F}(\mathcal{C} \otimes \mathcal{C})$  (where  $\mu \in \Gamma/\mathbb{Z}$  means  $\mu$  is a representative of the coset  $\mu + \mathbb{Z}$  in  $\Gamma$ , in case  $\mu \in \mathbb{Z}$  or  $\mu \in \Gamma/\mathbb{Z}$  we always choose  $\mu = 0$ ),

$$(2.4) \quad \begin{aligned} D_0(L_{\pm 1}) &= \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} a_{\mu,i}^{\pm} L_{\mu+i \pm 1} \otimes L_{-\mu-i} + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} b_{\mu,i}^{\pm} L_{\mu+i \pm 1} \otimes I_{-\mu-i} \\ &+ \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} c_{\mu,i}^{\pm} I_{\mu+i \pm 1} \otimes L_{-\mu-i} + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} d_{\mu,i}^{\pm} I_{\mu+i \pm 1} \otimes I_{-\mu-i} \\ &+ a^{\pm} L_{\pm 1} \otimes C + b^{\pm} C \otimes L_{\pm 1} + c^{\pm} I_{\pm 1} \otimes C + d^{\pm} C \otimes I_{\pm 1}, \end{aligned}$$

and we set  $b_{0,0}^{+} = c_{0,-1}^{+} = d_{0,0}^{+} = d_{0,-1}^{+} = b_{0,0}^{-} = c_{0,1}^{-} = d_{0,0}^{-} = d_{0,1}^{-} = 0$ , for some  $a_{\mu,i}^{\pm}, b_{\mu,i}^{\pm}, c_{\mu,i}^{\pm}, d_{\mu,i}^{\pm}, a^{\pm}$ ,

$b^\pm, c^\pm, d^\pm \in \mathbb{F}$ , where  $\{(\mu, i) \mid a_{\mu, i}^\pm \neq 0\}$ ,  $\{(\mu, i) \mid b_{\mu, i}^\pm \neq 0\}$ ,  $\{(\mu, i) \mid c_{\mu, i}^\pm \neq 0\}$  and  $\{(\mu, i) \mid d_{\mu, i}^\pm \neq 0\}$  are finite sets. In the following, to simplify notations, we shall always omit the superscript “+”; for example,  $a_{\mu, i} = a_{\mu, i}^+$ . Note that for  $\mu \in \Gamma/\mathbb{Z}$ ,  $i \in \mathbb{Z}$ , we have

$$(L_{\mu+i} \otimes L_{-\mu-i})_{inn}(L_1) = (\mu + i - 1)L_{\mu+i+1} \otimes L_{-\mu-i} - (\mu + i + 1)L_{\mu+i} \otimes L_{-\mu-i+1},$$

$$(L_{\mu+i} \otimes I_{-\mu-i})_{inn}(L_1) = (\mu + i - 1)L_{\mu+i+1} \otimes I_{-\mu-i} - (\mu + i)L_{\mu+i} \otimes I_{-\mu-i+1},$$

$$(I_{\mu+i} \otimes L_{-\mu-i})_{inn}(L_1) = (\mu + i)I_{\mu+i+1} \otimes L_{-\mu-i} - (\mu + i + 1)I_{\mu+i} \otimes L_{-\mu-i+1},$$

$$(I_{\mu+i} \otimes I_{-\mu-i})_{inn}(L_1) = (\mu + i)I_{\mu+i+1} \otimes I_{-\mu-i} - (\mu + i)I_{\mu+i} \otimes I_{-\mu-i+1},$$

$$(L_0 \otimes C)_{inn}(L_1) = -L_1 \otimes C,$$

$$(C \otimes L_0)_{inn}(L_1) = -C \otimes L_1.$$

Denote

$$M_\mu = \max\{|i| \mid a_{\mu, i} \neq 0\}, N_\mu = \max\{|i| \mid b_{\mu, i} \neq 0\},$$

$$E_\mu = \max\{|i| \mid c_{\mu, i} \neq 0\}, F_\mu = \max\{|i| \mid d_{\mu, i} \neq 0\}.$$

Using the above equations and the induction on  $M_\mu + N_\mu + E_\mu + F_\mu$ , by replacing  $D_0$  by  $D_0 - u_{inn}$ , where  $u$  is a combination of some  $L_{\mu+i} \otimes L_{-\mu-i}$ ,  $L_{\mu+i} \otimes I_{-\mu-i}$ ,  $I_{\mu+i} \otimes L_{-\mu-i}$ ,  $I_{\mu+i} \otimes I_{-\mu-i}$ , we can suppose

$$a = b = 0,$$

$$a_{\mu, i} = 0 \text{ if } \mu = 0; i \neq -2, 1, \text{ or } \mu \neq 0, i \neq 0,$$

$$b_{\mu, i} = 0 \text{ if } \mu = 0; i \neq -1, 1, \text{ or } \mu \neq 0, i \neq 0,$$

$$c_{\mu, i} = 0 \text{ if } \mu = 0; i \neq -2, 0, \text{ or } \mu \neq 0, i \neq 0,$$

$$d_{\mu, i} = 0 \text{ if } \mu = 0; i \neq -1, 0, \text{ or } \mu \neq 0, i \neq 0.$$

Thus we have, under modulo  $\mathbb{F}(C \otimes C)$ ,

$$\begin{aligned}
 D_0(L_1) = & a_{0, -2}L_{-1} \otimes L_2 + a_{0, 1}L_2 \otimes L_{-1} + \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} a_{\mu, i}L_{\mu+1} \otimes L_{-\mu} \\
 & + b_{0, -1}L_0 \otimes I_1 + b_{0, 1}L_2 \otimes I_{-1} + \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} b_{\mu, 0}L_{\mu+1} \otimes I_{-\mu} \\
 (2.5) \quad & + c_{0, -2}I_{-1} \otimes L_2 + c_{0, 0}I_1 \otimes L_0 + \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} c_{\mu, 0}I_{\mu+1} \otimes L_{-\mu} \\
 & + d_{0, -1}I_0 \otimes I_1 + d_{0, 0}I_1 \otimes I_0 + \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} d_{\mu, 0}I_{\mu+1} \otimes I_{-\mu} \\
 & + cI_1 \otimes C + dC \otimes I_1.
 \end{aligned}$$

Applying  $D_0$  to  $[L_{-1}, L_1] = 2L_0$ , we obtain, under modulo  $\mathbb{F}(\mathcal{C} \otimes \mathcal{C})$ ,

$$\begin{aligned}
& 3a_{0,-2}L_{-1} \otimes L_1 + 3a_{0,1}L_1 \otimes L_{-1} + \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} a_{\mu,0}[(\mu+2)L_\mu \otimes L_{-\mu} - (\mu-1)L_{\mu+1} \otimes L_{-\mu-1}] \\
& + b_{0,-1}[L_{-1} \otimes I_1 + L_0 \otimes I_0 + 2L_0 \otimes C_L] + b_{0,1}[3L_1 \otimes I_{-1} - L_2 \otimes I_{-2}] \\
& + \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} b_{\mu,0}[(\mu+2)L_\mu \otimes I_{-\mu} - \mu L_{\mu+1} \otimes I_{-\mu-1}] + c_{0,-2}[-I_{-2} \otimes L_2 + 3I_{-1} \otimes L_1] \\
& + c_{0,0}[I_0 \otimes L_0 + 2C_L \otimes L_0 + I_1 \otimes L_{-1}] + \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} c_{\mu,0}[(\mu+1)I_\mu \otimes L_{-\mu} - (\mu-1)I_{\mu+1} \otimes L_{-\mu-1}] \\
& + \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} d_{\mu,0}[(\mu+1)I_\mu \otimes I_{-\mu} - \mu I_{\mu+1} \otimes I_{-\mu-1}] \\
& = \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} a_{\mu,i}^- [(\mu+i-2)L_{\mu+i} \otimes L_{-\mu-i} - (\mu+i+1)L_{\mu+i-1} \otimes L_{-\mu-i+1}] \\
& + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} b_{\mu,i}^- [(\mu+i-2)L_{\mu+i} \otimes I_{-\mu-i} - (\mu+i)L_{\mu+i-1} \otimes I_{-\mu-i+1}] \\
& + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} c_{\mu,i}^- [(\mu+i-1)I_{\mu+i} \otimes L_{-\mu-i} - (\mu+i+1)I_{\mu+i-1} \otimes L_{-\mu-i+1}] \\
& + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} d_{\mu,i}^- [(\mu+i-1)I_{\mu+i} \otimes I_{-\mu-i} - (\mu+i)I_{\mu+i-1} \otimes I_{-\mu-i+1}] \\
& - 2a^- L_0 \otimes C - 2b^- C \otimes L_0.
\end{aligned}$$

Comparing the coefficients of  $L_{\mu+i} \otimes L_{-\mu-i}$  with  $\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}$ , we obtain

$$\begin{aligned}
& (\mu+i-2)a_{\mu,i}^- - (\mu+i+2)a_{\mu,i+1}^- \\
& = 3\delta_{\mu,0}\delta_{i,-1}a_{0,-2} + 3\delta_{\mu,0}\delta_{i,1}a_{0,1} + (1-\delta_{\mu,0})\delta_{i,0}(\mu+2)a_{\mu,0} + (1-\delta_{\mu,0})\delta_{i,1}(1-\mu)a_{\mu,0}.
\end{aligned}$$

Since  $\{(\mu, i) \mid a_{\mu,i}^- \neq 0\}$  is a finite set, fix  $\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}$ , we obtain

$$a_{0,m+1}^- = a_{0,-m}^- = a_{\mu,i}^- = 0 \quad \text{for } m \geq 2, \mu \neq 0, i \neq 1,$$

and we have the following relations:

(2.6)

$$a_{0,-1}^- = -a_{0,-2} - \frac{1}{3}a_{0,0}^-, a_{0,1}^- = -a_{0,0}^-, a_{0,2}^- = -a_{0,1} + \frac{1}{3}a_{0,0}^-, a_{\mu,1}^- = -a_{\mu,0} \quad \text{for } \mu \neq 0.$$

Comparing the coefficients of  $L_{\mu+i} \otimes I_{-\mu-i}$  with  $\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}$ , we obtain

$$\begin{aligned}
& (\mu+i-2)b_{\mu,i}^- - (\mu+i+1)b_{\mu,i+1}^- \\
& = \delta_{\mu,0}(\delta_{i,-1} + \delta_{i,0})b_{0,-1} + \delta_{\mu,0}(3\delta_{i,1} - \delta_{i,2})b_{0,1} + (1-\delta_{\mu,0})\delta_{i,0}(\mu+2)b_{\mu,0} - (1-\delta_{\mu,0})\delta_{i,1}\mu b_{\mu,0}.
\end{aligned}$$

Since  $\{(\mu, i) \mid b_{\mu,i}^- \neq 0\}$  is a finite set, fix  $\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}$ , and  $b_{0,0}^- = 0$ , we obtain

$$b_{0,-1} = b_{0,1} = b_{\mu,0} = b_{\mu,i}^- = b_{0,i}^- = 0 \quad \text{for } \mu \neq 0.$$

Comparing the coefficients of  $I_{\mu+i} \otimes L_{-\mu-i}$  with  $\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}$ , we obtain

$$\begin{aligned}
& (\mu+i-1)c_{\mu,i}^- - (\mu+i+2)c_{\mu,i+1}^- \\
& = \delta_{\mu,0}(-\delta_{i,-2} + 3\delta_{i,-1})c_{0,-2} + \delta_{\mu,0}(\delta_{i,1} + \delta_{i,0})c_{0,0} + (1-\delta_{\mu,0})\delta_{i,0}(\mu+1)c_{\mu,0} + (1-\delta_{\mu,0})\delta_{i,1}(1-\mu)c_{\mu,0}.
\end{aligned}$$

Since  $\{(\mu, i) \mid c_{\mu,i}^- \neq 0\}$  is a finite set, fix  $\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}$ , and  $c_{0,1}^- = 0$ , we obtain

$$c_{0,-2} = c_{0,0} = c_{\mu,0} = c_{\mu,i}^- = c_{0,i}^- = 0 \quad \text{for } \mu \neq 0.$$

Comparing the coefficients of  $I_{\mu+i} \otimes I_{-\mu-i}$  with  $\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}$ , we obtain

$$\begin{aligned} & (\mu + i - 1)d_{\mu,i}^- - (\mu + i + 1)c_{\mu,i+1}^- \\ &= (1 - \delta_{\mu,0})\delta_{i,0}(\mu + 1)d_{\mu,0} - (1 - \delta_{\mu,0})\delta_{i,1}\mu d_{\mu,0}. \end{aligned}$$

Since  $\{(\mu, i) \mid d_{\mu,i}^- \neq 0\}$  is a finite set, fix  $\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}$ , and  $d_{0,0}^- = d_{0,-1}^- = 0$ , we obtain

$$d_{\mu,m}^- = d_{0,i}^- = 0 \quad \text{for } m \neq 1, \mu \neq 0,$$

and we have the following relation:

$$(2.7) \quad d_{\mu,1}^- = -d_{\mu,0} \quad \text{for } \mu \neq 0.$$

Comparing the coefficients of  $L_0 \otimes C, C \otimes L_0$ , we obtain

$$a^- = b^- = 0.$$

Consequently, under modulo  $\mathbb{F}(\mathcal{C} \otimes \mathcal{C})$ , we can rewrite

$$\begin{aligned} D_0(L_{-1}) &= a_{0,-1}^- L_{-2} \otimes L_1 + a_{0,0}^- L_{-1} \otimes L_0 + a_{0,1}^- L_0 \otimes L_{-1} + a_{0,2}^- L_1 \otimes L_{-2} \\ (2.8) \quad & - \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} a_{\mu,0} L_\mu \otimes L_{-\mu-1} - \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} d_{\mu,0} I_\mu \otimes I_{-\mu-1} + c^- I_{-1} \otimes C \\ & + d^- C \otimes I_{-1}, \end{aligned}$$

where the coefficients satisfy (2.6) and (2.7).

Under modulo  $\mathbb{F}(\mathcal{C} \otimes \mathcal{C})$ , we can write

$$\begin{aligned} D_0(L_{\pm 2}) &= \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} a_{\mu,i}'^\pm L_{\mu+i\pm 2} \otimes L_{-\mu-i} + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} b_{\mu,i}'^\pm L_{\mu+i\pm 2} \otimes I_{-\mu-i} \\ (2.9) \quad & + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} c_{\mu,i}'^\pm I_{\mu+i\pm 2} \otimes L_{-\mu-i} + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} d_{\mu,i}'^\pm I_{\mu+i\pm 2} \otimes I_{-\mu-i} \\ & + a'^\pm L_{\pm 2} \otimes C + b'^\pm C \otimes L_{\pm 2} + c'^\pm I_{\pm 2} \otimes C + d'^\pm C \otimes I_{\pm 2}, \end{aligned}$$

and we set  $b_{0,0}'^+ = c_{0,-2}'^+ = d_{0,0}'^+ = d_{0,-2}'^+ = b_{0,0}'^- = c_{0,2}'^- = d_{0,0}'^- = d_{0,2}'^- = 0$ , for some  $a_{\mu,i}'^\pm, b_{\mu,i}'^\pm, c_{\mu,i}'^\pm, d_{\mu,i}'^\pm, a'^\pm, b'^\pm, c'^\pm, d'^\pm \in \mathbb{F}$ , where  $\{(\mu, i) \mid a_{\mu,i}'^\pm \neq 0\}, \{(\mu, i) \mid b_{\mu,i}'^\pm \neq 0\}, \{(\mu, i) \mid c_{\mu,i}'^\pm \neq 0\}$  and  $\{(\mu, i) \mid d_{\mu,i}'^\pm \neq 0\}$  are finite sets. Now we shall omit the superscript “+” again; for example,  $a_{\mu,i}'^+ = a_{\mu,i}'$ .

Applying  $D_0$  to  $[L_{-1}, L_2] = 3L_1$ , we have, under modulo  $\mathbb{F}(\mathcal{C} \otimes \mathcal{C})$ ,

$$\begin{aligned}
& \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} a'_{\mu,i} [(\mu+i+3)L_{\mu+i+1} \otimes L_{-\mu-i} - (\mu+i-1)L_{\mu+i+2} \otimes L_{-\mu-i-1}] \\
& + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} b'_{\mu,i} [(\mu+i+3)L_{\mu+i+1} \otimes I_{-\mu-i} - (\mu+i)L_{\mu+i+2} \otimes I_{-\mu-i-1}] \\
& + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} c'_{\mu,i} [(\mu+i+2)I_{\mu+i+1} \otimes L_{-\mu-i} - (\mu+i-1)I_{\mu+i+2} \otimes L_{-\mu-i-1}] \\
& + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} d'_{\mu,i} [(\mu+i+2)I_{\mu+i+1} \otimes I_{-\mu-i} - (\mu+i)I_{\mu+i+2} \otimes I_{-\mu-i-1}] \\
& + 3a' L_1 \otimes C + 3b' C \otimes L_1 + 2c' I_1 \otimes C + 2d' C \otimes I_1 + a_{0,-1}^- [4L_0 \otimes L_1 + L_{-2} \otimes L_3] \\
& + a_{0,0}^- [3L_1 \otimes L_0 + 2L_{-1} \otimes L_2] + a_{0,1}^- [2L_2 \otimes L_{-1} + 3L_0 \otimes L_1] + a_{0,2}^- [L_3 \otimes L_{-2} + 4L_1 \otimes L_0] \\
& + \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} a_{\mu,0}^- [(\mu-2)L_{\mu+2} \otimes L_{-\mu-1} - (\mu+3)L_{\mu} \otimes L_{-\mu-3}] \\
& + \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} d_{\mu,0}^- [\mu I_{\mu+2} \otimes I_{-\mu-1} - (\mu+1)I_{\mu} \otimes I_{-\mu+1}] + c^- I_1 \otimes C + d^- C \otimes I_1 \\
& = 3a_{0,-2} L_{-1} \otimes L_2 + 3a_{0,1} L_2 \otimes L_{-1} + \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} 3a_{\mu,0} L_{\mu+1} \otimes L_{-\mu} \\
& + \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} 3d_{\mu,0} I_{\mu+1} \otimes I_{-\mu} + 3c I_1 \otimes C + 3d C \otimes I_1.
\end{aligned}$$

Comparing the coefficients of  $L_{\mu+i+1} \otimes L_{-\mu-i}$  with  $\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}$ , we obtain

$$\begin{aligned}
3\delta_{i,-2}a_{0,-2} + 3\delta_{i,1}a_{0,1} = & (i+3)a'_{0,i} - (i-2)a'_{0,i-1} + \delta_{i,-3}a_{0,-1}^- + 2\delta_{i,-2}a_{0,0}^- \\
& + \delta_{i,-1}(4a_{0,-1}^- + 3a_{0,1}^-) + \delta_{i,0}(4a_{0,2}^- + 3a_{0,0}^{-(L^{-1})}) + 2\delta_{i,1}a_{0,1}^- + \delta_{i,2}a_{0,2}^-,
\end{aligned}$$

and

$$(\mu+i+3)a'_{\mu,i} - (\mu+i-2)a'_{\mu,i-1} + \delta_{i,1}(\mu-2)a_{\mu,0} - \delta_{i,-1}(\mu+3)a_{\mu,0} = 3\delta_{i,0}a_{\mu,0} \quad \text{for } \mu \neq 0.$$

Using (2.6), we obtain

$$a_{0,-1}^- = a_{0,2}^- = a'_{0,m} = a'_{0,-m-2} = a_{\mu,0} = a'_{\mu,i} = 0 \quad \text{for } m \geq 2, \mu \neq 0, i \in \mathbb{Z},$$

and

$$\begin{aligned}
(2.10) \quad & a_{0,0}^- = -a_{0,1}^- = 3a_{0,1} = -3a_{0,-2}, \\
& a'_{0,-3} = a'_{0,1} - 6a_{0,1}, a'_{0,-2} = -4a'_{0,1} + 15a_{0,1}, \\
& a'_{0,-1} = 6a'_{0,1} - 18a_{0,1}, a'_{0,0} = -4a'_{0,1} + 9a_{0,1}.
\end{aligned}$$

Comparing the coefficients of  $L_{\mu+i+1} \otimes I_{-\mu-i}$  with  $\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}$ , we obtain

$$b'_{\mu,i} = 0.$$

Comparing the coefficients of  $I_{\mu+i+1} \otimes L_{-\mu-i}$  with  $\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}$ , we obtain

$$c'_{\mu,i} = 0.$$

Comparing the coefficients of  $I_{\mu+i+1} \otimes I_{-\mu-i}$  with  $\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}$ , we obtain

$$(i+2)d'_{0,i} - (i-1)d'_{0,i-1} = 0,$$

and



$$(\mu + i + 2)d'_{\mu,i} - (\mu + i - 1)d'_{\mu,i-1} + \mu\delta_{i,1}d_{\mu,0} - (\mu + 1)\delta_{i,-1}d_{\mu,0} = 3\delta_{i,0}d_{\mu,0} \quad \text{for } \mu \neq 0.$$

Using  $d'_{0,0} = d'_{0,-2} = 0$ , we obtain

$$d'_{0,i} = d'_{\mu,m} = d'_{\mu,-m-1} = 0 \quad \text{for } m \geq 1, \mu \neq 0, i \in \mathbb{Z},$$

and

$$(2.11) \quad d'_{\mu,-1} = d'_{\mu,0} = d_{\mu,0}, \quad \text{for } \mu \neq 0.$$

Now (2.5) and (2.8), respectively, become, under modulo  $\mathbb{F}(\mathcal{C} \otimes \mathcal{C})$ ,

$$(2.12) \quad \begin{aligned} D_0(L_1) = & a_{0,1}(-L_{-1} \otimes L_2 + L_2 \otimes L_{-1}) \\ & + \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} d_{\mu,0} I_{\mu+1} \otimes I_{-\mu} + c I_1 \otimes C + d C \otimes I_1, \end{aligned}$$

$$(2.13) \quad \begin{aligned} D_0(L_{-1}) = & 3a_{0,1}(L_{-1} \otimes L_0 - L_0 \otimes L_{-1}) \\ & - \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} d_{\mu,0} I_{\mu} \otimes I_{-\mu-1} + c^- I_{-1} \otimes C + d^- C \otimes I_{-1}. \end{aligned}$$

Applying  $D_0$  to  $[L_{-2}, L_1] = 3L_{-1}$ , we have, under modulo  $\mathbb{F}(\mathcal{C} \otimes \mathcal{C})$ ,

$$\begin{aligned} & a_{0,1}[-L_{-3} \otimes L_2 - 4L_{-1} \otimes L_0 - \tfrac{1}{2}L_{-1} \otimes C_L + 4L_0 \otimes L_{-1} + \tfrac{1}{2}C_L \otimes L_{-1} + L_2 \otimes L_{-3}] \\ & + \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} d_{\mu,0}[(\mu + 1)I_{\mu-1} \otimes I_{-\mu} - \mu I_{\mu+1} \otimes I_{-\mu-2}] + c I_{-1} \otimes C + d C \otimes I_{-1} \\ & - (\sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} a'_{\mu,i} [(\mu + i - 3)L_{\mu+i-1} \otimes L_{-\mu-i} - (\mu + i + 1)L_{\mu+i-2} \otimes L_{-\mu-i+1}]) \\ & + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} b'_{\mu,i} [(\mu + i - 3)L_{\mu+i-1} \otimes I_{-\mu-i} - (\mu + i)L_{\mu+i-2} \otimes I_{-\mu-i+1}] \\ & + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} c'_{\mu,i} [(\mu + i - 2)I_{\mu+i-1} \otimes L_{-\mu-i} - (\mu + i + 1)I_{\mu+i-2} \otimes L_{-\mu-i+1}] \\ & + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} d'_{\mu,i} [(\mu + i - 2)I_{\mu+i-1} \otimes I_{-\mu-i} - (\mu + i)I_{\mu+i-2} \otimes I_{-\mu-i+1}] \\ & - 3a'_{-} L_{-1} \otimes C - 3b'_{-} C \otimes L_{-1} - 2c'_{-} I_{-1} \otimes C - 2d'_{-} C \otimes I_{-1} \\ & = 9a_{0,1}(L_{-1} \otimes L_0 - L_0 \otimes L_{-1}) - 3 \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} d_{\mu,0} I_{\mu} \otimes I_{-\mu-1} + 3c^- I_{-1} \otimes C + 3d^- C \otimes I_{-1}. \end{aligned}$$

Comparing the coefficients, Using  $d'_{0,0} = d'_{0,2} = 0$ , we obtain

$$a_{0,1} = a'_{\mu,i} = a'_{0,m+2} = a'_{0,-m} = a'_{-} = 0 \quad \text{for } m \geq 2, \mu \neq 0, i \in \mathbb{Z},$$

$$b'_{\mu,i} = c'_{\mu,i} = b'_{-} = d'_{0,i} = 0 \quad \text{for } \mu \neq 0, i \in \mathbb{Z},$$

$$d'_{\mu,m} = d'_{\mu,-m+3} = 0 \quad \text{for } m \geq 3, \mu \neq 0,$$

and

$$a'_{0,0} = a'_{0,2} = -4a'_{0,-1}, a'_{0,1} = 6a'_{0,-1}, a'_{0,3} = a'_{0,-1},$$

$$(2.14) \quad d'_{\mu,1} = d'_{\mu,2} = -d_{\mu,0} \quad \text{for } \mu \neq 0.$$

Now (2.12), (2.13), and (2.9), respectively, become, under modulo  $\mathbb{F}(\mathcal{C} \otimes \mathcal{C})$ ,

$$(2.15) \quad D_0(L_1) = \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} d_{\mu,0} I_{\mu+1} \otimes I_{-\mu} + c I_1 \otimes C + d C \otimes I_1,$$

$$(2.16) \quad D_0(L_{-1}) = - \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} d_{\mu,0} I_{\mu} \otimes I_{-\mu-1} + c^- I_{-1} \otimes C + d^- C \otimes I_{-1},$$

$$(2.17) \quad \begin{aligned} D_0(L_2) = & a'_{0,1} (L_{-1} \otimes L_3 - 4L_0 \otimes L_2 + 6L_1 \otimes L_1 - 4L_2 \otimes L_0 + L_3 \otimes L_{-1}) \\ & + \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} d_{\mu,0} (I_{\mu+2} \otimes I_{-\mu} + I_{\mu+1} \otimes I_{-\mu+1}) + c' I_2 \otimes C + d' C \otimes I_2, \end{aligned}$$

$$(2.18) \quad \begin{aligned} D_0(L_{-2}) = & a'^-_{0,-1} (L_{-3} \otimes L_1 - 4L_{-2} \otimes L_0 + 6L_{-1} \otimes L_{-1} - 4L_0 \otimes L_{-2} + L_1 \otimes L_{-3}) \\ & - \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} d_{\mu,0} (I_{\mu-1} \otimes I_{-\mu-1} + I_{\mu} \otimes I_{-\mu-2}) + c'^- I_{-2} \otimes C + d'^- C \otimes I_{-2}. \end{aligned}$$

Note that

$$\begin{aligned} (I_{\mu-1} \otimes I_{-\mu+1})_{inn}(L_2) &= (\mu-1) I_{\mu+1} \otimes I_{-\mu+1} - (\mu-1) I_{\mu-1} \otimes I_{-\mu+3}, \\ (I_{\mu-3} \otimes I_{-\mu+3})_{inn}(L_2) &= (\mu-3) I_{\mu-1} \otimes I_{-\mu+3} - (\mu-3) I_{\mu-3} \otimes I_{-\mu+5}. \end{aligned}$$

Using these two equations, by replacing  $D_0$  by  $D_0 - u_{inn}$ , where  $u$  is a combination of  $I_{\mu-1} \otimes I_{-\mu+1}$  and  $I_{\mu-3} \otimes I_{-\mu+3}$  (this replacement does not affect the above equations (2.15), (2.16) and (2.18)), under modulo  $\mathbb{F}(\mathcal{C} \otimes \mathcal{C})$ , we can rewrite (2.17) as

$$(2.19) \quad \begin{aligned} D_0(L_2) = & a'_{0,1} (L_{-1} \otimes L_3 - 4L_0 \otimes L_2 + 6L_1 \otimes L_1 - 4L_2 \otimes L_0 + L_3 \otimes L_{-1}) \\ & + \sum_{0 \neq \mu \in \Gamma/\mathbb{Z}} d_{\mu,0} (I_{\mu+2} \otimes I_{-\mu} + I_{\mu-3} \otimes I_{-\mu+5}) + c' I_2 \otimes C + d' C \otimes I_2, \end{aligned}$$

Applying  $D_0$  to  $[L_{-2}, L_2] = L_0 + \frac{1}{2}C_L$ , we obtain

$$a'_{0,1} = -a'^-_{0,-1}, d_{\mu,0} = 0 \quad \text{for } \mu \neq 0.$$

Using (2.7), (2.11) and (2.14), we obtain

$$d_{\mu,1}^- = d'_{\mu,-1} = d'_{\mu,0} = d'^-_{\mu,1} = d'^-_{\mu,2} = 0 \quad \text{for } \mu \neq 0.$$

Denote

$$u = L_{-1} \otimes L_1 - 2L_0 \otimes L_0 + L_1 \otimes L_{-1}.$$

Replacing  $D_0$  by  $D_0 + a'_{0,1} u_{inn}$  (this replacement does not affect the above two equations (2.15) and (2.16)), we obtain

$$(2.20) \quad D_0(L_2) = c' I_2 \otimes C + d' C \otimes I_2,$$

$$(2.21) \quad D_0(L_{-2}) = c'^- I_{-2} \otimes C + d'^- C \otimes I_{-2}.$$

At last, we can rewrite (2.4) and (2.9) as

$$(2.22) \quad D_0(L_{\pm 1}) = c^{\pm} I_{\pm 1} \otimes C + d^{\pm} C \otimes I_{\pm 1},$$

$$(2.23) \quad D_0(L_{\pm 2}) = c'^{\pm} I_{\pm 2} \otimes C + d'^{\pm} C \otimes I_{\pm 2}.$$

Write under modulo  $\mathbb{F}(\mathcal{C} \otimes \mathcal{C})$ ,

$$\begin{aligned}
 D_0(I_{\pm 1}) = & \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} e_{\mu,i}^{\pm} L_{\mu+i \pm 1} \otimes L_{-\mu-i} + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} f_{\mu,i}^{\pm} L_{\mu+i \pm 1} \otimes I_{-\mu-i} \\
 (2.24) \quad & + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} g_{\mu,i}^{\pm} I_{\mu+i \pm 1} \otimes L_{-\mu-i} + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} h_{\mu,i}^{\pm} I_{\mu+i \pm 1} \otimes I_{-\mu-i} \\
 & + e^{\pm} L_{\pm 1} \otimes C + f^{\pm} C \otimes L_{\pm 1} + g^{\pm} I_{\pm 1} \otimes C + h^{\pm} C \otimes I_{\pm 1},
 \end{aligned}$$

and we set  $f_{0,0}^{+} = g_{0,-1}^{+} = h_{0,0}^{+} = h_{0,-1}^{+} = f_{0,0}^{-} = g_{0,1}^{-} = h_{0,0}^{-} = h_{0,1}^{-} = 0$ , for some  $e_{\mu,i}^{\pm}, f_{\mu,i}^{\pm}, g_{\mu,i}^{\pm}, h_{\mu,i}^{\pm}$ ,  $e^{\pm}, f^{\pm}, g^{\pm}, h^{\pm} \in \mathbb{F}$ , where  $\{(\mu, i) \mid e_{\mu,i}^{\pm} \neq 0\}$ ,  $\{(\mu, i) \mid f_{\mu,i}^{\pm} \neq 0\}$ ,  $\{(\mu, i) \mid g_{\mu,i}^{\pm} \neq 0\}$  and  $\{(\mu, i) \mid h_{\mu,i}^{\pm} \neq 0\}$  are finite sets. In the following, to simplify notations, we shall always omit the superscript “+”; for example,  $e_{\mu,i} = e_{\mu,i}^{+}$ . Note that for  $\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}$ , we have

$$(L_0 \otimes C)_{inn}(I_1) = -I_1 \otimes C,$$

$$(C \otimes L_0)_{inn}(I_1) = -C \otimes I_1.$$

Denote  $M'_{\mu} = \max\{|i| \mid e_{\mu,i} \neq 0\}$ ,  $N'_{\mu} = \max\{|i| \mid f_{\mu,i} \neq 0\}$ ,  $E'_{\mu} = \max\{|i| \mid g_{\mu,i} \neq 0\}$ ,  $F'_{\mu} = \max\{|i| \mid h_{\mu,i} \neq 0\}$ .

Thus we have, under modulo  $\mathbb{F}(\mathcal{C} \otimes \mathcal{C})$ ,

$$\begin{aligned}
 D_0(I_1) = & \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} e_{\mu,i} L_{\mu+i+1} \otimes L_{-\mu-i} + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} f_{\mu,i} L_{\mu+i+1} \otimes I_{-\mu-i} \\
 (2.25) \quad & + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} g_{\mu,i} I_{\mu+i+1} \otimes L_{-\mu-i} + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} h_{\mu,i} I_{\mu+i+1} \otimes I_{-\mu-i} \\
 & + e L_1 \otimes C + f C \otimes L_1
 \end{aligned}$$

Applying  $D_0$  to  $[L_{-1}, I_1] = 0$ , we obtain

$$(2.26) \quad D_0(I_1) = 0,$$

By the same method, we can obtain

$$(2.27) \quad D_0(I_{-1}) = 0.$$

Using (2.22), (2.23), (2.26) and (2.27), under modulo  $\mathbb{F}(\mathcal{C} \otimes \mathcal{C})$ , we have

$$(2.28) \quad D_0(L_i) = c_i I_i \otimes C + d_i C \otimes I_i \quad \text{for } i \in \mathbb{Z},$$

$$(2.29) \quad D_0(I_i) = 0 \quad \text{for } i \in \mathbb{Z}.$$

For a fixed  $0 \neq \mu \in \Gamma/\mathbb{Z}$ , Under modulo  $\mathbb{F}(\mathcal{C} \otimes \mathcal{C})$ , we can write

$$\begin{aligned}
 D_0(L_{\mu}) = & \sum_{\nu \in \Gamma/\mathbb{Z}, j \in \mathbb{Z}} a_{\nu,j} L_{\mu+\nu+j} \otimes L_{-\nu-j} + \sum_{\nu \in \Gamma/\mathbb{Z}, j \in \mathbb{Z}} b_{\nu,j} L_{\mu+\nu+j} \otimes I_{-\nu-j} \\
 (2.30) \quad & + \sum_{\nu \in \Gamma/\mathbb{Z}, j \in \mathbb{Z}} c_{\nu,j} I_{\mu+\nu+j} \otimes L_{-\nu-j} + \sum_{\nu \in \Gamma/\mathbb{Z}, j \in \mathbb{Z}} d_{\nu,j} I_{\mu+\nu+j} \otimes I_{-\nu-j} \\
 & + a_{\mu} L_{\mu} \otimes C + b_{\mu} C \otimes L_{\mu} + c_{\mu} I_{\mu} \otimes C + d_{\mu} C \otimes I_{\mu},
 \end{aligned}$$

for some  $a_{\mu,j}, b_{\mu,j}, c_{\mu,j}, d_{\mu,j} \in \mathbb{F}$ , where  $A' = \{(\nu, j) \mid a_{\nu,j} \neq 0\}$ ,  $B' = \{(\nu, j) \mid b_{\nu,j} \neq 0\}$ ,  $C' = \{(\nu, j) \mid c_{\nu,j} \neq 0\}$ ,  $D' = \{(\nu, j) \mid d_{\nu,j} \neq 0\}$  are finite sets. Let  $i \gg 0$ . Applying  $D_0$  to  $[L_i, [L_{-i}, L_\mu]] = (\mu + i)(\mu - 2i)L_\mu$ , we obtain

$$(2.31) \quad \begin{aligned} L_i \cdot L_{-i} \cdot D_0(L_\mu) &= (\mu + i)(\mu - 2i)D_0(L_\mu) + L_i \cdot L_\mu \cdot D_0(L_{-i}) \\ &\quad + (\mu + i)L_{\mu-i} \cdot D_0(L_i) \end{aligned}$$

Define the total order on  $\Gamma \times \Gamma$  by

$$(\alpha, \beta) > (\alpha', \beta') \Leftrightarrow \alpha > \alpha' \text{ or } \alpha = \alpha', \beta > \beta'.$$

If  $A' \neq \emptyset$ , let  $(\nu_0, j_0)$  be the maximal element in  $A'$ , in this case,  $L_{\mu+\nu_0+j_0+i} \otimes L_{-\nu_0-j_0-i}$  is the leading term of  $L_i \cdot L_{-i} \cdot D_0(L_\mu)$ , a contradiction to equation (2.29). Similarly, we have  $B' = C' = D' = \emptyset$ . Thus, we can rewrite (2.30) as under modulo  $\mathbb{F}(\mathcal{C} \otimes \mathcal{C})$ ,

$$D_0(L_\mu) = a_\mu L_\mu \otimes C + b_\mu C \otimes L_\mu + c_\mu I_\mu \otimes C + d_\mu^{(L_\mu)} C \otimes I_\mu.$$

Then we can denote  $u = \frac{a_\mu}{\mu} L_0 \otimes C + \frac{b_\mu}{\mu} C \otimes L_0$ , replacing  $D_0$  by  $D_0 + u_{inn}$ , we obtain

$$(2.32) \quad D_0(L_\mu) = c_\mu I_\mu \otimes C + d_\mu C \otimes I_\mu.$$

Similarly, we have  $D_0(I_\nu) = e_\nu L_\nu \otimes C + f_\nu C \otimes L_\nu$ .

Applying  $D_0$  to  $[L_{\nu-1}, I_1] = I_\nu (0 \neq \nu \in \Gamma/\mathbb{Z})$ , we obtain, under modulo  $\mathbb{F}(\mathcal{C} \otimes \mathcal{C})$ ,  $D_0(I_\nu) = 0$ . Using (2.28) and (2.32), we have  $D_0(L_\mu) = c_\mu I_\mu \otimes C + d_\mu C \otimes I_\mu$  for  $\mu \in \Gamma \setminus \{0\}$ , applying  $D_0$  to  $[L_\mu, L_\nu] = (\nu - \mu)L_{\mu+\nu}$ , we obtain  $c_\mu, d_\mu \in \mathbb{F}$ . Claim 4 is proved.

*Claim 5..* We can suppose  $D_0 = 0$  by replacing  $D_0$  by  $D_0 - u_{inn} - (\lambda \otimes C + C \otimes \eta)$  for some  $u \in V_0, \lambda, \eta \in \mathbb{F}$ .

Replacing  $D_0$  by  $D_0 - u_{inn} - (\lambda \otimes C + C \otimes \eta)$  for some  $u \in V_0, \lambda, \eta \in \mathbb{F}$ , the above claims have proved  $D_0(\mathfrak{L}) \subset \mathbb{F}(C \otimes C)$ . Because  $\mathfrak{L} = [\mathfrak{L}, \mathfrak{L}]$ , we obtain  $D_0(\mathfrak{L}) \subset \mathfrak{L} \cdot D_0(\mathfrak{L}) = 0$ .

*Claim 6..* For every  $D \in \text{Der}(\mathfrak{L}, V)$ , (2.2) is a finite sum.

According to the above results, for any  $\alpha \in \Gamma$ , we can suppose  $D_\alpha = (u_\alpha)_{inn} + \lambda \otimes C + C \otimes \eta$  for some  $u_\alpha \in V_\alpha, \lambda, \eta \in \mathbb{F}$ . If  $\{\alpha \mid u_\alpha \neq 0\}$  is an infinite set, then, we obtain  $D(L_0) = \sum_{\alpha \in \Gamma} L_0 \cdot u_\alpha = \sum_{\alpha \in \Gamma} \alpha u_\alpha + \lambda \otimes C + C \otimes \eta$  is an infinite sum, a contradiction with the fact that  $D$  is a derivation from  $\mathfrak{L} \rightarrow V$ . This proves the claim and Proposition 2.4.

Now, we can complete the proof of Theorem 2.2 (1) as follows. First we need

*Lemma 2.5..* Suppose  $r \in V(\text{mod } \mathbb{F}(\mathcal{C} \otimes \mathcal{C}))$  such that  $\omega \cdot r \in \text{Im}(1 - \tau)$  for all  $\omega \in \mathfrak{L}$ . Then  $r \in \text{Im}(1 - \tau)$ .

*Proof.* First note that  $\mathfrak{L} \cdot \text{Im}(1 - \tau) \subset \text{Im}(1 - \tau)$ . Write  $r = \sum_{\alpha \in \Gamma} r_\alpha$  with  $r_\alpha \in V_\alpha$ . Obviously,

$$(2.33) \quad r \in \text{Im}(1 - \tau) \iff r_\alpha \in \text{Im}(1 - \tau) \text{ for all } \alpha \in \Gamma.$$

Thus without loss of generality, we can suppose  $r = r_\alpha$  is homogeneous.

For any  $\alpha \neq 0$ , then  $r_\alpha = \frac{1}{\alpha} L_0 \cdot r_\alpha \in \text{Im}(1 - \tau)$ . Thus assume  $\alpha = 0$ . Now we can write

$$\begin{aligned}
 (2.34) \quad r_0 = & \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} a_{\mu,i} L_{\mu+i} \otimes L_{-\mu-i} + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} b_{\mu,i} L_{\mu+i} \otimes I_{-\mu-i} \\
 & + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} c_{\mu,i} I_{\mu+i} \otimes L_{-\mu-i} + \sum_{\mu \in \Gamma/\mathbb{Z}, i \in \mathbb{Z}} d_{\mu,i} I_{\mu+i} \otimes I_{-\mu-i} \\
 & + a_\mu L_0 \otimes C + b_\mu C \otimes L_0 + c_\mu I_0 \otimes C + d_\mu C \otimes I_0,
 \end{aligned}$$

for some  $a_{\mu,i}, b_{\mu,i}, c_{\mu,i}, d_{\mu,i} \in \mathbb{F}$ , where  $\{(\mu, i) \mid a_{\mu,i} \neq 0\}$ ,  $\{(\mu, i) \mid b_{\mu,i} \neq 0\}$ ,  $\{(\mu, i) \mid c_{\mu,i} \neq 0\}$ ,  $\{(\mu, i) \mid d_{\mu,i} \neq 0\}$  are finite sets. For a fixed  $\mu \in \Gamma/\mathbb{Z}$ , let  $H_1 = \max\{|i| \mid a_{\mu,i} \neq 0\}$ ,  $H_2 = \min\{|i| \mid a_{\mu,i} \neq 0\}$ . We call  $L_{\mu+H_1} \otimes L_{-\mu-H_1}$  and  $L_{\mu+H_2} \otimes L_{-\mu-H_2}$  respectively the highest and the lowest terms of  $r_0$ .

Choose  $k \in \mathbb{Z}$ ,  $k \neq H_1, -H_2$ . Then,  $L_{k+\mu+H_1} \otimes L_{-\mu-H_1}$  and  $L_{\mu+H_2} \otimes L_{-\mu-H_2+k}$  are the highest and the lowest terms of  $L_k \cdot r_0$  with coefficients  $(\mu + H_1 - k)a_{\mu,H_1}$  and  $(-\mu - H_2 - k)a_{\mu,H_2}$  respectively. Since  $L_k \cdot r_0 \in \text{Im}(1 - \tau)$ , we must have  $H_1 = -H_2$  and  $a_{\mu,H_1} = -b_{\mu,H_2}$ . Induction on  $H_1$  shows  $a_{\mu,i} = -a_{\mu,-i}$  for all  $i \in \mathbb{Z}$ . Similarly, we can obtain  $b_{\mu,i} = -b_{\mu,-i}, c_{\mu,i} = -c_{\mu,-i}, d_{\mu,i} = -d_{\mu,-i}$  for all  $i \in \mathbb{Z}$ , and  $a_\mu = -b_\mu, c_\mu = -d_\mu$ . Thus  $r_0 \in \text{Im}(1 - \tau)$ . This proves the lemma.

*Proof of Theorem 2.2(1)* Let  $(\mathfrak{L}, [\cdot, \cdot], \Delta)$  be a Lie bialgebra structure on  $\mathfrak{L}$ . By (1.14), Definition 1.1 and Proposition(2.4),  $\Delta = \Delta_r + \sigma$ , where  $r \in V(\text{mod}(\mathcal{C} \otimes \mathcal{C}))$  and  $\sigma \in \mathbb{F} \otimes \mathbb{F} C + \mathbb{F} C \otimes \mathbb{F}$ . By (1.4)  $\text{Im} \Delta \subset \text{Im}(1 - \tau)$ , so  $\Delta_r(L_\alpha) + \sigma(L_\alpha) \in \text{Im}(1 - \tau)$  for  $\alpha \in \Gamma$ , which implies that  $c_\mu = -d_\mu$ . Moreover,  $\Delta_r(I_\alpha) \in \text{Im}(1 - \tau)$  for  $\alpha \in \Gamma$ . Thus,  $\sigma(\mathfrak{L}) \in \text{Im}(1 - \tau)$ . So  $\text{Im} \Delta_r \in \text{Im}(1 - \tau)$ . It follows immediately from Lemma 2.5 that  $r \in \text{Im}(1 - \tau)(\text{mod}(\mathcal{C} \otimes \mathcal{C}))$ . According to the above results, we always have  $0 \neq \lambda \in \mathbb{F}$  make  $\sigma = \lambda \otimes C - C \otimes \lambda$ . Then for any  $\omega_\alpha \in \mathfrak{L}_\alpha, \alpha \in \Gamma$ , we have

$$\begin{aligned}
 & (1 + \xi + \xi^2) \cdot (1 \otimes \sigma) \cdot \sigma(\omega_\alpha) \\
 & = \lambda(1 - \delta_{\alpha,0})w_1(1 + \xi + \xi^2)(1 \otimes \sigma)(I_\alpha \otimes C - C \otimes I_\alpha) \\
 & = 0.
 \end{aligned}$$

It shows  $(\mathfrak{L}, [\cdot, \cdot], \sigma)$  is a Lie bialgebra, and the proof of Theorem 2.2(1) is completed. But by Proposition (2.4), there is no  $0 \neq r \in \mathfrak{L}$ , such that  $\Delta = x \cdot r$ . Thus,  $(\mathfrak{L}, [\cdot, \cdot], \Delta)$  can not be a coboundary Lie bialgebra.

### 3. LIE BIALGEBRAS OF THE CENTERLESS GENERALIZED HEISENBERG-VIRASORO ALGEBRA

Denote  $\overline{\mathfrak{L}} = \mathfrak{L}/\mathcal{C}$ , then  $\overline{\mathfrak{L}}$  is the centerless generalized Heisenberg-Virasoro algebra.

*Theorem 3.1.* (1) Every Lie bialgebra structure on the Lie algebra  $\overline{\mathfrak{L}}$  is a triangular coboundary Lie bialgebra.

(2) An element  $r \in \text{Im}(1 - \tau) \subset \overline{\mathfrak{L}} \otimes \overline{\mathfrak{L}}$  satisfies CYBE if and only if it satisfies MYBE.

(3) Regarding  $\overline{V} = \overline{\mathfrak{L}} \otimes \overline{\mathfrak{L}}$  as an  $\overline{\mathfrak{L}}$ -module under the adjoint diagonal action of  $\overline{\mathfrak{L}}$ , we have  $H^1(\overline{\mathfrak{L}}, \overline{V}) = \text{Der}(\overline{\mathfrak{L}}, \overline{V})/\text{Inn}(\overline{\mathfrak{L}}, \overline{V}) = 0$ .

*Proof* According to the Theorem 2.2(2)(3),  $\text{Der}(\overline{\mathfrak{L}}, \overline{V})/\text{Inn}(\overline{\mathfrak{L}}, \overline{V}) = 0$ . So (2),(3) hold obviously. By Theorem 2.2(1), we obtain  $\Delta = \Delta_r, r \in \text{Im}(1 - \tau)$ . Then by Lemma 1.3, we have  $c(r) = 0$ . Thus,  $(\overline{\mathfrak{L}}, [\cdot, \cdot], \Delta)$  is a triangular coboundary Lie bialgebra. Theorem is proved.  $\square$

## REFERENCES

1. V.G. Drinfeld, Hamiltonian structures on Lie group, Lie algebras and the geometric meaning of classical Yang-Baxter equations, *Soviet Math. Dokl.* **27**(1) (1983), 68-71.
2. V.G. Drinfeld, Quantum groups, in: *Proceeding of the International Congress of Mathematicians*, Vol. 1, 2, Berkeley, Calif. 1986, Amer. Math. Soc., Providence, RI, 1987, pp. 798-820.
3. D. Liu, Y. Pei, L. Zhu, Lie bialgebra structures on the twisted Heisenberg-Virasoro algebra, *J. Alg.*, 359(2012), 35-48.
4. D. Liu, L. Chen, L. Zhu, Lie superbialgebra structures on the  $N = 2$  superconformal algebra, *J. Geom. Phys.*, 62(2012), 826-831.
5. D. Liu, L. Zhu, Generalized Heisenberg-Virasoro algebras, *Frontiers of Mathematics in China* 2009, 1673-3452.
6. J. Han, J. Li, Y. Su, Lie bialgebras of the Schdinger-Virasoro algebra, *J. Math. Phys.*, 50(8)(2009)pp. 083504-083504-12.
7. W. Michaelis, A class of infinite-dimensional Lie bialgebras containing the Virasoro algebras, *Adv. Math.* **107** (1994), 365-392.
8. S.-H. Ng, E.J. Taft, Classification of the Lie bialgebra structures on the Witt and Virasoro algebras, *J. Pure Appl. Algebra* **151** (2000), 67-88.
9. R. Shen, Q. Jiang, Y. Su, Verma modules over the generalized Heisenberg-Virasoro algebra, *Comm. Alg.*, 36(4) (2008), 1464-1473.
10. G. Song, Y. Su, Lie bialgebras of generalized Witt type, *Science in China: Series A*, 49(2006), 533-544.
11. E.J. Taft, Witt and Virasoro algebras as Lie bialgebras, *J. Pure Appl. Algebra* **87** (1993), 301-312.
12. Y. Wu, G. Song, Y. Su, Lie bialgebras of generalized Virasoro-like type, *Acta Mathematica Sinica, English Series*, 22(2006), 1915-1922.
13. Y. Wu, G. Song, Y. Su, Lie bialgebras of generalizd Witt type II. *Commun Algebra*, 2007, **3**: 1992-2007.
14. H. Yang, Y. Su, Lie bialgebras over the Ramond  $N = 2$  super-Virasoro algebra, *Chaos Solitons Fractals*, 40(2)(2009), 661-671.
15. X. Yue, Y. Su, Lie bialgebra structures on Lie algebras of generalized Weyl type, *Comm. Alg.*, 36(4)(2008), 1537-1549.
16. X. Yue, G. Song, Y. Su, Lie bialgebra structures on Lie algebras of generalized weyl type, *Comm. Algebra*, 36(4)(2008), 1537-1549.